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Transmission and Reflection of Waves in a One-Dimensional Disordered Array

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We study wave propagation in a one-dimensional disordered array of scattering potentials. We calculate the mean and the variance of the resistance of the array, defined as the ratio of reflected to transmitted intensity, for a rather wide class of probability distributions characterizing the disorder. Our method is based on a mapping of the wave propagation onto the motion of a two-dimensional oscillator which is perturbed parametrically.

KEY WORDS: Waves in disordered array; one dimension; conductivity; parametrically perturbed oscillator; exponential growth.

1. INTRODUCTION

The problem of wave propagation in disordered one-dimensional arrays of scatterers has received a great deal of attention. We refer to reviews by Ishii[(1)] and by Erdös and Herndon[(2)]. It was first argued by Landauer[(3)] that for the case of electrons in a disordered one-dimensional lattice, described by the Schrödinger equation with randomly placed potentials, the electrical resistivity is proportional to the ratio of reflected to transmitted intensity of an incident plane wave. This dimensionless quantity is called, for brevity, the resistance. He showed, on the basis of some simplifying assumptions, that the resistance grows exponentially with the number of obstacles, rather than linearly. Exact calculations have confirmed this result[(2,4)] although the growth factor calculated by Landauer turns out to be valid only in the classical high wavenumber limit.

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Erdös and Herndon[2] have developed an elegant method for calculating both the mean and the variance of the resistance for a fixed number of scatterers in a wide class of models. It is of interest to calculate the variance as well as the mean, because the distribution of resistance broadens rapidly with increasing number of scatterers.[5] The method of Erdös and Herndon[2] is based on a transfer matrix formalism which leads to a multiple product of 4×4 matrices for the calculation of the mean and to a product of 16×16 matrices for the calculation of the variance. Group theoretical arguments are invoked to reduce the size of the matrices to 3×3 and 5×5, respectively.

We show here that the calculation of the mean and variance of the resistance may be simplified considerably. Our method is based on a mapping of the wave propagation onto the motion of a two-dimensional oscillator which is perturbed parametrically. The resistance is proportional to the total increase in energy of the oscillator due to the perturbations. The formulation leads directly to matrices of reduced size and to explicit formulas for the mean and variance of the resistance for a wide class of probability distributions of scatterer configurations. The calculation is extended to the case where N scatterers are distributed over a fixed length L.

2. TRANSFER MATRICES

We consider the time-independent Schrödinger equation describing wave propagation through a one-dimensional disordered array of scatterers. With appropriate transcription the calculation applies to classical problems, such as acoustic or electromagnetic wave propagation. The time-independent Schrödinger equation reads

$$-D \frac{d^2\varphi}{dx^2} + \sum_{j=1}^{N} V_j(x - x_j)\, \varphi(x) = E\varphi(x) \tag{2.1}$$

where $D = \hbar^2/2m$ and E is the energy. We assume that the scattering potentials do not overlap, are ordered $x_1 < x_2 < \cdots < x_N$, and that their statistics are governed by a known probability distribution. The effect of a single scatterer on an incident plane wave may be described by a transfer matrix relating the amplitudes a_+ and a_- of the wavefunction $\varphi(x) = a_+ \exp(ikx) + a_- \exp(-ikx)$ to the left of the scatterer to the amplitudes a'_+ and a'_- of the wavefunction $\varphi(x) = a'_+ \exp(ikx) + a'_- \exp(-ikx)$ to the right by

$$\begin{pmatrix} a'_+ \\ a'_- \end{pmatrix} = \mathbf{R}_j \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{2.2}$$

The 2×2 matrix $\mathbf{R}_j$ may be written as[2]

$$\mathbf{R}_j = \mathbf{U}_j^* \mathbf{M}_j \mathbf{U}_j \tag{2.3}$$

with the diagonal matrix

$$\mathbf{U}_j = \begin{pmatrix} \exp(ikx_j) & 0 \\ 0 & \exp(-ikx_j) \end{pmatrix} \tag{2.4}$$

and with a matrix $\mathbf{M}_j$ which is independent of position. The location x_j of the scatterer may be defined such that $\mathbf{M}_j$ has the parametric representation

$$\mathbf{M}_j = \begin{pmatrix} \exp(-i\gamma_j) \cosh \delta_j & -i \sinh \delta_j \\ i \sinh \delta_j & \exp(i\gamma_j) \cosh \delta_j \end{pmatrix} \tag{2.5}$$

with real parameters γ_j and δ_j which depend on energy. If the potential $V_j(x - x_j)$ is symmetric, then x_j coincides with the center of symmetry. More generally the location x_j depends on the energy E. We shall choose coordinates such that the center x_1 of the first scatterer is located at the origin $x = 0$. Clearly this may require a shift of origin dependent on the energy.

The transmission through the array of a wave incident from the left is described by the transfer relation

$$\begin{pmatrix} T \\ 0 \end{pmatrix} = \mathbf{W}(N) \begin{pmatrix} 1 \\ R \end{pmatrix} \tag{2.6}$$

where T is the transmission coefficient and R is the reflection coefficient. The transfer matrix

$$\mathbf{W}(N) = \begin{pmatrix} \dfrac{1}{T^*} & -\dfrac{R^*}{T^*} \\ -\dfrac{R}{T} & \dfrac{1}{T} \end{pmatrix} \tag{2.7}$$

is given by the product

$$\mathbf{W}(N) = \mathbf{U}_N^* \mathbf{M}_N \mathbf{G}(x_N - x_{N-1}) \, \mathbf{M}_{N-1} \cdots \mathbf{M}_2 \mathbf{G}(x_2) \, \mathbf{M}_1 \tag{2.8}$$

with the propagation matrix

$$\mathbf{G}(\xi_j) = \begin{pmatrix} \exp(ik\xi_j) & 0 \\ 0 & \exp(-ik\xi_j) \end{pmatrix} \qquad \xi_j = x_j - x_{j-1} \tag{2.9}$$

We shall be interested in the statistical distribution of the resistance ρ defined by $\rho = |R|^2/|T|^2$.

3. CLASSICAL MOTION PICTURE

We shall find it convenient to use the correspondence of the wave propagation described above to the motion of a classical harmonic oscillator with parametric perturbation. One finds an evident correspondence to the motion of a one-dimensional oscillator by identifying the wavefunction φ with the coordinate q of the oscillator, D with the mass, x with the time, and $k=\sqrt{E/D}$ with the frequency. We remark that the motion of such an oscillator with added damping has been studied extensively in the theory of stochastic processes.[6-10] However, a more useful correspondence to the motion of a two-dimensional oscillator may be obtained in the following manner. We choose two real standard solutions $q_1(x)$ and $q_2(x)$ of the differential equation (2.1) with the properties

$$q_1(x)=\cos kx \qquad q_2(x)=\sin kx \qquad x\ll 0 \tag{3.1}$$

for x sufficiently far to the left of the first scatterer. We regard $q_1(x)$ and $q_2(x)$, which are real for all x, as the components of the position $\mathbf{q}(x)$ of a two-dimensional oscillator. The frequency of the oscillator is perturbed parametrically as described by the potentials $V_j(x-x_j)$. We put $t=kx$ so that the momentum $\mathbf{p}=(p_1, p_2)$ is given by the equations

$$p_1(x)=k^{-1}\frac{dq_1}{dx} \qquad p_2(x)=k^{-1}\frac{dq_2}{dx} \tag{3.2}$$

It is clear from (3.1) that $(\mathbf{q}(x), \mathbf{p}(x))$ is identified with the motion of the two-dimensional oscillator for special initial conditions. As usual it will be convenient to use instead of real coordinates and momenta the complex amplitudes

$$a_1=q_1+ip_1 \qquad a_2=q_2+ip_2 \tag{3.3}$$

It follows from the transfer matrix formalism of the preceding section that the motion $(\mathbf{q}(x), \mathbf{p}(\mathrm{x}))$ may be mapped uniquely onto the motion of an oscillator whose position and momentum change instantaneously by hits occurring at instants $x_1,..., x_N$ and which oscillates harmonically between hits. Outside the range of the potential the two motions are identical and it suffices to consider the second one. The harmonic motion provides a linear relation between coordinates and momenta $(\mathbf{q}, \mathbf{p})$ just after the hit at x_{j-1} to those just before the hit at x_j

$$\begin{aligned} \mathbf{q}' &= c_j\mathbf{q}+s_j\mathbf{p} \\ \mathbf{p}' &= -s_j\mathbf{q}+c_j\mathbf{p} \end{aligned} \tag{3.4}$$

with the abbreviations

$$c_j = \cos k\xi_j \qquad s_j = \sin k\xi_j \tag{3.5}$$

where $\xi_j = x_j - x_{j-1}$. In terms of the complex amplitudes (3.3) this relation may be expressed as

$$\begin{pmatrix} a_i' \\ (a_i^*)' \end{pmatrix} = \mathbf{G}(\xi_j) \begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \qquad i = 1, 2 \tag{3.6}$$

where the matrix $\mathbf{G}(\xi_j)$ is given by (2.9). The effect of the hit occurring at x_j is described by a linear relation between coordinates and momenta $(\mathbf{q}', \mathbf{p}')$ before the hit to $(\mathbf{q}'', \mathbf{p}'')$ after the hit

$$\begin{aligned} \mathbf{q}'' &= u_j \mathbf{q}' + v_j \mathbf{p}' \\ \mathbf{p}'' &= w_j \mathbf{q}' + u_j \mathbf{p}' \end{aligned} \tag{3.7}$$

with coefficients

$$\begin{aligned} &u_j = \cos \gamma_j \cosh \delta_j \qquad v_j = \sinh \delta_j - \sin \gamma_j \cosh \delta_j \\ &w_j = \sinh \delta_j + \sin \gamma_j \cosh \delta_j \end{aligned} \tag{3.8}$$

which are related by

$$u_j^2 - v_j w_j = 1 \tag{3.9}$$

In terms of the complex amplitudes (3.3) the collision law (3.7) is expressed as

$$\begin{pmatrix} a_i'' \\ (a_i^*)'' \end{pmatrix} = \mathbf{M}_j \begin{pmatrix} a_i' \\ (a_i^*)' \end{pmatrix} \qquad i = 1, 2 \tag{3.10}$$

where the matrix $\mathbf{M}_j$ is given by (2.5).

During the harmonic motion, both $p_1^2 + q_1^2$ and $p_2^2 + q_2^2$ are conserved quantities, but these energies are not conserved at a hit, as may be seen from (3.7). However, there is a quantity which is conserved both during the harmonic motion and during hits. This quantity may be identified with the angular momentum of the two-dimensional oscillator. With the choice of initial conditions (3.1) one has

$$q_1 p_2 - q_2 p_1 = 1 \tag{3.11}$$

or in terms of the complex amplitudes (3.3)

$$a_1^* a_2 - a_1 a_2^* = 2i \tag{3.12}$$

In mathematical terms the quantity (3.11) is just the Wronskian of the two independent solutions of the differential equation (2.1). In quantum mechanical language the conservation law (3.11) corresponds to the constant probability current density.

4. RESISTANCE

In this section we express the resistance $\rho = |R|^2/|T|^2$ in terms of the classical variables $(\mathbf{q}, \mathbf{p})$ defined in the preceding section. To this purpose we write the scattering solution of (2.1) corresponding to a plane wave incident from the left as a linear combination of the two standard solutions defined in (3.1)

$$\varphi_s(x) = (1+R)\, q_1(x) + i(1-R)\, q_2(x) \tag{4.1}$$

Beyond the last scatterer this solution must equal the transmitted wave

$$\varphi_s(x) = T\exp(ikx), \qquad x > x_N \tag{4.2}$$

We choose a point X which is to the right of x_N for all members of the ensemble of scatterer configurations. Equating (4.1) and (4.2) at this point we obtain

$$(1+R)\, q_1(X) + i(1-R)\, q_2(X) = T\exp(ikX) \tag{4.3}$$

Similarly by equating the derivatives at X

$$(1+R)\, p_1(X) + i(1-R)\, p_2(X) = iT\exp(ikX) \tag{4.4}$$

Solving for R and T from (4.3) and (4.4) we find

$$\begin{aligned} R &= -\frac{a_1(X) + ia_2(X)}{a_1(X) - ia_2(X)} \\ T &= \frac{2}{a_1(X) - ia_2(X)} \exp(-ikX) \end{aligned} \tag{4.5}$$

Hence the resistance ρ is given by the simple expression[3]

$$\rho = \tfrac{1}{2}\varepsilon(X) - \tfrac{1}{2} \tag{4.6}$$

where $\varepsilon(X)$ is the energy of the two-dimensional oscillator after the last hit

$$\varepsilon(X) = \tfrac{1}{2}[a_1^*(X)\, a_1(X) + a_2^*(X)\, a_2(X)] \tag{4.7}$$

[3] This analogy was worked out in collaboration with Professor G. W. Ford.

In deriving (4.6) we have made use of (3.12). From (3.1) it follows that the initial value of $|a_1|^2 + |a_2|^2$ equals unity. It is known that on average the resistance ρ grows exponentially with the number of hits. In the following we show how the mean $\langle \rho \rangle$ and the variance $\langle (\rho - \langle \rho \rangle)^2 \rangle$ may be evaluated for a class of models.

5. PROPER COLLISION VARIABLES

It is clear from (4.6) that in order to find the resistance ρ for a given configuration of scatterers we may study equivalently how the energy of the oscillator changes in a sequence of hits. Since the harmonic motion conserves the energy it suffices to study the collision process in more detail. Dropping a prime on either side in (3.10) and omitting the subscript j specifying the event we write the collision law in the abbreviated form

$$\begin{pmatrix} a_i' \\ (a_i^*)' \end{pmatrix} = \begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha^* \end{pmatrix} \begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \qquad i = 1, 2 \tag{5.1}$$

where the complex α and real β are related by

$$|\alpha|^2 - \beta^2 = 1 \tag{5.2}$$

and are given explicitly by (2.5). It would clearly be sufficient to write the expression for a_i' since the one for $(a_i^*)'$ then follows by complex conjugation. However the form (5.1) is of interest since it shows explicitly how the pair (a_i, a_i^*) transforms into the pair $[a_i', (a_i^*)']$ upon collision. We shall call the pair (a_i, a_i^*) a proper collision variable, or for brevity a proper 2 vector. The transformation matrix of the 2 vector is just the transfer matrix $\mathbf{M}$, but in this connection we shall call it the two-dimensional collision matrix and denote it by $\mathbf{K}_2$. According to (3.6) between collisions the 2 vector transforms with the propagation matrix $\mathbf{G}$ defined in (2.9). In this connection we shall denote it as $\mathbf{G}_2$. We repeat the explicit form of the two matrices

$$\mathbf{K}_2 = \begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha^* \end{pmatrix} \qquad \mathbf{G}_2 = \begin{pmatrix} e^{ik\xi} & 0 \\ 0 & e^{-ik\xi} \end{pmatrix} \tag{5.3}$$

It is of interest to look for higher order proper collision variables. Thus we find by straightforward calculation the proper 3 vector $(a_i^* a_j^*, a_i^* a_j + a_i a_j^*, a_i a_j)$ which upon collisions transforms as

$$\begin{pmatrix} (a_i^* a_j^*)' \\ (a_i^* a_j + a_i a_j^*)' \\ (a_i a_j)' \end{pmatrix} = \mathbf{K}_3 \begin{pmatrix} a_i^* a_j^* \\ a_i^* a_j + a_i a_j^* \\ a_i a_j \end{pmatrix} \qquad (i, j) = 1, 2 \tag{5.4}$$

with the three-dimensional collision matrix

$$\mathbf{K}_3 = \begin{pmatrix} \alpha^2 & -i\alpha\beta & -\beta^2 \\ 2i\alpha\beta & |\alpha|^2+\beta^2 & -2i\alpha^*\beta \\ -\beta^2 & i\alpha^*\beta & \alpha^{*2} \end{pmatrix} \tag{5.5}$$

The corresponding propagation matrix $\mathbf{G}_3$ is given by

$$\mathbf{G}_3 = \begin{pmatrix} e^{2ik\xi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2ik\xi} \end{pmatrix} \tag{5.6}$$

In the same manner one finds a proper 5 vector which transforms upon collisions as

$$\begin{pmatrix} (a_i^{*2}a_j^{*2})' \\ (a_i^{*2}a_j^*a_j + a_i^*a_ia_j^{*2})' \\ (a_i^{*2}a_j^2 + 4a_i^*a_ia_j^*a_j + a_i^2a_j^{*2})' \\ (a_i^*a_ia_j^2 + a_i^2a_j^*a_j)' \\ (a_i^2a_j^2)' \end{pmatrix} = \mathbf{K}_5 \begin{pmatrix} a_i^{*2}a_j^{*2} \\ a_i^{*2}a_j^*a_j + a_i^*a_ia_j^{*2} \\ a_i^{*2}a_j^2 + 4a_i^*a_ia_j^*a_j + a_i^2a_j^{*2} \\ a_i^*a_ia_j^2 + a_i^2a_j^*a_j \\ a_i^2a_j^2 \end{pmatrix} \tag{5.7}$$

where again $(i, j) = 1, 2$ and where the five-dimensional collision matrix $\mathbf{K}_5$ is given by

$$\mathbf{K}_5 = \begin{pmatrix} \alpha^4 & -2i\alpha^3\beta & -\alpha^2\beta^2 & 2i\alpha\beta^3 & \beta^4 \\ 2i\alpha^3\beta & \alpha^2(|\alpha|^2+\beta^2) & -i\alpha\beta(|\alpha|^2+\beta^2) & -\beta^2(3\,|\alpha|^2+\beta^2) & 2i\alpha^*\beta^3 \\ -6\alpha^2\beta^2 & 6i\alpha\beta(|\alpha|^2+\beta^2) & |\alpha|^4+4\,|\alpha|^2\,\beta^2+\beta^4 & -6i\alpha^*\beta(|\alpha|^2+\beta^2) & -6\alpha^{*2}\beta^2 \\ -2i\alpha\beta^3 & -\beta^2(3\,|\alpha|^2+\beta^2) & i\alpha^*\beta(|\alpha|^2+\beta^2) & \alpha^{*2}(|\alpha|^2+3\beta^2) & -2i\alpha^{*3}\beta \\ \beta^4 & -2i\alpha^*\beta^3 & -\alpha^{*2}\beta^2 & 2i\alpha^{*3}\beta & \alpha^{*4} \end{pmatrix} \tag{5.8}$$

The corresponding propagation matrix $\mathbf{G}_5$ is diagonal and is given by

$$\mathbf{G}_5 = \begin{pmatrix} e^{4ik\xi} & & & & \\ & e^{2ik\xi} & & 0 & \\ & & 1 & & \\ & 0 & & e^{-2ik\xi} & \\ & & & & e^{-4ik\xi} \end{pmatrix} \tag{5.9}$$

The above matrices are all we need for the discussion of the mean and the variance of the resistance ρ. To conclude this section we note that by taking the real and imaginary parts of the relations (5.4) and (5.7) one obtains real proper collision variables expressed in terms of the ps and qs which transform with a real matrix. However, the complex formulation is more convenient because for the complex variables the propagation matrix is diagonal.

6. PROBABILITY DISTRIBUTIONS

We are now in a position to choose the systems for which the mean and variance of the resistance may be evaluated. We restrict ourselves to models for which the properties of individual scatterers and the distance between scatterers are not correlated. Furthermore, we assume that the probability distribution of the scattering parameters (γ_j, δ_j) is the same for all scatterers and that the probability distribution of scattering centers is governed by a nearest neighbor pair distribution which is the same for all pairs. Thus we consider a probability distribution of the form

$$P(x_1, \gamma_1, \delta_1,\ldots, x_N, \gamma_N, \delta_N) = \delta(x_1) \prod_{j=2}^{N} f(x_j - x_{j-1}) \prod_{j=1}^{N} h(\gamma_j, \delta_j) \quad (6.1)$$

where both $f(\xi)$ and $h(\gamma, \delta)$ are normalized to unity

$$\int_0^{\infty} f(\xi)\, d\xi = 1 \qquad \int_0^{2\pi} \int_{-\infty}^{\infty} h(\gamma, \delta)\, d\gamma\, d\delta = 1 \quad (6.2)$$

As special cases one may consider models where $f(\xi) = \delta(\xi - l)$, so that the scatterers are fixed at lattice sites and there is only site disorder,[11,12] or where $h(\gamma, \delta) = \delta(\gamma - \gamma_0)\, \delta(\delta - \delta_0)$, so that all scatterers are identical and there is only distance disorder. We recall that for an asymmetric scatterer the definition of the center x_j may depend on energy. This case is permitted as long as the center is the same for all scatterers included in the probability distribution $h(\gamma_j, \delta_j)$. Averages over either $f(\xi)$ or $h(\gamma, \delta)$ will be denoted by pointed brackets $\langle \cdots \rangle$, whereas the average over the complete probability distribution (6.1) will be denoted by $\langle \cdots \rangle_N$.

We shall also wish to average over a subensemble of N scatterers with fixed total length L. The corresponding probability distribution is given by

$$\begin{aligned} &P_L(x_1, \gamma_1, \delta_1,\ldots, x_N, \gamma_N, \delta_N) \\ &\quad = \delta(x_1)\, \delta(x_N - L) \prod_{j=2}^{N} f(x_j - x_{j-1}) \prod_{j=1}^{N} h(\gamma_j, \delta_j)/F_N(L) \end{aligned} \quad (6.3)$$

where the length distribution $F_N(L)$ is given by

$$F_N(L) = \langle \delta(x_N - L) \rangle_N = \langle \delta(L - \sum_{j=2}^{N} \xi_j) \rangle_N \tag{6.4}$$

The Laplace transform of the length distribution is

$$\hat{F}_N(s) = \int_0^\infty e^{-sL} F_N(L)\, dL = (\hat{f}(s))^{N-1} \tag{6.5}$$

where $\hat{f}(s)$ is the Laplace transform of the neighbor distribution

$$\hat{f}(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi, \qquad \hat{f}(0) = 1 \tag{6.6}$$

Averages over the probability distribution (6.3) will be denoted by $\langle \cdots \rangle_{N,L}$. Evaluation of such an average will involve an inverse Laplace transform of a quantity $\langle \cdots \rangle_{N,s}$ calculated at fixed N and s.

7. MEAN AND VARIANCE OF THE RESISTANCE

Using the probability distributions described above we can now evaluate the mean and the variance of the resistance. We recall that the resistance ρ is given by the expression (4.6). We define the unit 3 vector $\mathbf{e}_2 = (0, 1, 0)$. Using the initial conditions (3.1) we easily find that for a fixed configuration of N scatterers the energy in (4.7) is given by the matrix element

$$\varepsilon(X) = (\mathbf{e}_2 | \mathbf{K}_3(N)\, \mathbf{G}_3(\xi_N) \cdots \mathbf{G}_3(\xi_2)\, \mathbf{K}_3(1) | \mathbf{e}_2) \tag{7.1}$$

where the matrices $\mathbf{K}_3$ and $\mathbf{G}_3$ are given by (5.5) and (5.6). Averaging over the probability distribution (6.1) we hence find

$$\langle \varepsilon \rangle_N = (\mathbf{e}_2 | \langle \mathbf{K}_3 \rangle (\langle \mathbf{G}_3 \rangle \langle \mathbf{K}_3 \rangle)^{N-1} | \mathbf{e}_2) \tag{7.2}$$

Similarly, multiplying the distribution (6.3) by the normalization factor $F_N(L)$, then taking the Laplace transform as in (6.5) and averaging we find

$$\langle \varepsilon \rangle_{N,s} = (\mathbf{e}_2 | \langle \mathbf{K}_3 \rangle (\langle \hat{\mathbf{G}}_3(s) \rangle \langle \mathbf{K}_3 \rangle)^{N-1} | \mathbf{e}_2) \tag{7.3}$$

where the matrix $\langle \hat{\mathbf{G}}_3(s) \rangle$ is defined by

$$\langle \hat{\mathbf{G}}_3(s) \rangle = \int_0^\infty e^{-s\xi} f(\xi)\, \mathbf{G}_3(\xi)\, d\xi \tag{7.4}$$

We recover the result (7.2) by putting $s=0$ in (7.3). The average at fixed N and L is now obtained as

$$\langle \varepsilon \rangle_{N,L} = \frac{1}{2\pi i F_N(L)} \int e^{sL} \langle \varepsilon \rangle_{N,s}\, ds \tag{7.5}$$

where the integration path goes from $-i\infty$ to $+i\infty$ in the complex s plane to the right of all singularities of the integrand.

Next we consider the variance of the resistance. We define the unit 5 vector $\mathbf{f}_3 = (0, 0, 1, 0, 0)$. Again it is easily shown that for a fixed configuration of N scatterers the square of the energy at the end of the sequence of hits is given by

$$\varepsilon^2(X) = \tfrac{2}{3}(\mathbf{f}_3 \,|\, \mathbf{K}_5(N)\, \mathbf{G}_5(\xi_N) \cdots \mathbf{G}_5(\xi_2)\, \mathbf{K}_5(1) \,|\, \mathbf{f}_3) + \tfrac{1}{3} \tag{7.6}$$

where use has been made of the conservation law (3.12). The matrices $\mathbf{K}_5$ and $\mathbf{G}_5$ are given by (5.8) and (5.9). Averaging over the probability distribution (6.1) we find

$$\langle \varepsilon^2 \rangle_N = \tfrac{2}{3}(\mathbf{f}_3 \,|\, \langle \mathbf{K}_5 \rangle (\langle \mathbf{G}_5 \rangle \langle \mathbf{K}_5 \rangle)^{N-1} \,|\, \mathbf{f}_3) + \tfrac{1}{3} \tag{7.7}$$

Similarly we find from (6.3)

$$\langle \varepsilon^2 - \tfrac{1}{3} \rangle_{N,s} = \tfrac{2}{3}(\mathbf{f}_3 \,|\, \langle \mathbf{K}_5 \rangle (\langle \hat{\mathbf{G}}_5(s) \rangle \langle \mathbf{K}_5 \rangle)^{N-1} \,|\, \mathbf{f}_3) \tag{7.8}$$

where the matrix $\langle \hat{\mathbf{G}}_5(s) \rangle$ is defined in analogy to (7.4). The average at fixed N and L is obtained as

$$\langle \varepsilon^2 \rangle_{N,L} = \frac{1}{3} + \frac{1}{2\pi i F_N(L)} \int e^{sL} \left\langle \varepsilon^2 - \frac{1}{3} \right\rangle_{N,s} ds \tag{7.9}$$

with integration path in the complex s plane from $-i\infty$ to $i\infty$ to the right of all singularities of the integrand.

8. EXPONENTIAL GROWTH

By specialization of the result (7.1) to a single scatterer we see that the energy of the two-dimensional oscillator after one hit has changed from unity to $|\alpha|^2 + \beta^2$, which according to (5.2) is always larger than 1. Hence in contrast to the individual components $\frac{1}{2}a_1^* a_1$ and $\frac{1}{2}a_2^* a_2$ the sum always increases after the first hit. It follows from (2.7), (3.10), and (5.1) that the factor $|\alpha|^2 + \beta^2$ may be expressed in terms of the reflection coefficient r of a single scatterer by

$$|\alpha|^2 + \beta^2 = \frac{1 + |r|^2}{1 - |r|^2} \tag{8.1}$$

Neglect of phase relations between successive scatterers, which is correct in the high wavenumber limit, implies that in the matrix $\langle \mathbf{G}_3 \rangle$ we replace the 11 and 33 elements by zero. Upon substitution in (7.2) only the Nth power of the 22 element of $\langle \mathbf{K}_3 \rangle$ survives and we obtain Landauer's result for the average resistance[3]

$$\langle \rho \rangle_N \approx \frac{1}{2} \left(\left\langle \frac{1+|r|^2}{1-|r|^2} \right\rangle \right)^N - \frac{1}{2} \tag{8.2}$$

Similarly it follows from (7.6) that after the first hit the square of the energy increases by a factor $(|\alpha|^2 + \beta^2)^2$. If we again neglect phase relations between successive scatterers and replace all elements in $\langle \mathbf{G}_5 \rangle$ but the 33 element by zero then we find in analogy to the above

$$\langle \varepsilon^2 \rangle_N \approx \frac{2}{3} \left(\left\langle \frac{1+4\,|r|^2+|r|^4}{(1-|r|^2)^2} \right\rangle \right)^N + \frac{1}{3} \tag{8.3}$$

This shows that in this approximation the variance of the resistance

$$\langle \rho^2 \rangle_N - \langle \rho \rangle_N^2 = \tfrac{1}{4} \langle \varepsilon^2 \rangle_N - \tfrac{1}{4} \langle \varepsilon \rangle_N^2 \tag{8.4}$$

grows exponentially with a larger exponent than the mean.

Of course (8.2) and (8.3) constitute only an approximation to the correct result. In fact the average $\langle \rho \rangle_N$ may even show oscillations as a function of N, as may be seen explicitly in the examples treated by Erdös and Herndon[2] in their Figs. 14, 15, and 20. The complete calculation requires evaluation of a power of the three-dimensional matrix $\langle \mathbf{G}_3 \rangle \langle \mathbf{K}_3 \rangle$ for the mean of the resistance and of a power of the five-dimensional matrix $\langle \mathbf{G}_5 \rangle \langle \mathbf{K}_5 \rangle$ for the variance. This may be achieved by standard methods. Interesting examples have been investigated by Erdös and Herndon.[2] Here we merely reproduce the explicit expression for the cubic equation which determines the growth rate of the mean resistance $\langle \rho \rangle_N$. From (7.1) we find that this is given by the characteristic equation $|\lambda \mathbf{I} - \langle \mathbf{G}_3 \rangle \langle \mathbf{K}_3 \rangle| = 0$ which reads explicitly

$$\lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0 \tag{8.5}$$

where the coefficients are given by

$$\begin{aligned} c_0 &= [(\langle \beta^2 \rangle^2 - |\langle \alpha^2 \rangle|^2)\langle |\alpha|^2 + \beta^2 \rangle + 4\mathrm{Re}(\langle \alpha^2 \rangle \langle \alpha^2 \beta \rangle^2) \\ &\quad - 4\,|\langle \alpha\beta \rangle|^2 \langle \beta^2 \rangle]\,|\chi(2k)|^2 \\ c_1 &= (|\langle \alpha^2 \rangle|^2 - \langle \beta^2 \rangle^2)\,|\chi(2k)|^2 + 2\langle |\alpha|^2 + \beta^2 \rangle\,\mathrm{Re}(\langle \alpha^2 \rangle\,\chi(2k)) \\ &\quad - 4\mathrm{Re}(\langle \alpha\beta \rangle^2\,\chi(2k)) \\ c_2 &= -\langle |\alpha|^2 + \beta^2 \rangle - 2\mathrm{Re}(\langle \alpha^2 \rangle\,\chi(2k)) \end{aligned} \tag{8.6}$$

with $\chi(2k)$ given by

$$\chi(2k) = \hat{f}(-2ik) = \int_0^\infty e^{2ik\xi} f(\xi)\, d\xi \tag{8.7}$$

The growth rate of $\langle \rho \rangle_N$ is given by $\ln \lambda_1$, where λ_1 is the largest positive real root of (8.5).

9. DISCUSSION

We have shown that the mapping of the wave propagation in a disordered array onto the motion of a two-dimensional oscillator with parametric perturbation provides a convenient method of studying the first problem. We remark that for the calculation of the mean resistance it suffices to study a one-dimensional oscillator. However, it follows from the expression (4.6) relating the resistance to the energy of the oscillator that for the study of the variance and of higher order moments the extension to two dimensions is essential. We presume that the method may be used with success in the more general context of stochastic differential equations.

In an ensemble where both the number of scatterers N and the length L of the sample are fixed one can evaluate the mean resistance $\langle \rho \rangle_{N,L}$ and its variance and study the dependence on N at constant L. It has been suggested[2,13] that $\langle \rho \rangle_{N,L}$ grows only with the exponential of $\sqrt{N}$, whereas the relative variance decreases with the exponential of $-N^{1/4}$. We hope to study this matter analytically in a succeeding article in which we shall apply the results (7.5) and (7.9) to a system of δ-function scatterers.

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